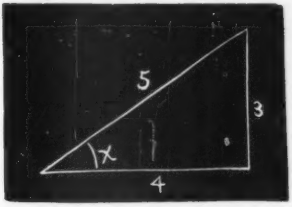
should be taught their use before proceeding further. Having given the value of a trigonometric function it is now easy to construct geometrically all the angles which satisfy the given value, and to find the values of the other five functions. Or, we may express any five of the functions in terms of the sixth. If the construction of the angle is not required, the process may be shortened by simply drawing a right triangle to serve as a check on the numerical part (not the algebraic signs) of the work. Thus having given $\sec x = \frac{5}{4}$, to find the value of the other five functions. Here

$$\sec x = \frac{5}{4} = \frac{\text{hypt.}}{\text{adj. leg}}$$



$$\text{opp. leg} = \sqrt{(5^2 - 4^2)} = 3.$$

The numerical values of the functions will then be:

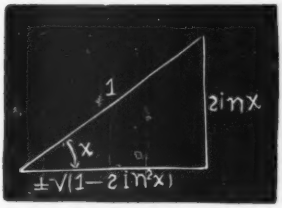
$$\sin x = \tfrac{3}{5} \qquad \csc x = \tfrac{5}{3}$$
$$\cos x = \tfrac{4}{5} \qquad \sec x = \tfrac{5}{4}$$
$$\tan x = \tfrac{3}{4} \qquad \cot x = \tfrac{4}{3}$$

The given secant being positive, the angle will lie either in the first or the fourth quadrants. If the angle lies in the first quadrant all the functions are positive and the above results are correct as they stand. If the angle lies in the fourth quadrant, we merely change the algebraic signs as follows:

$$\sin x = -\tfrac{3}{5} \qquad \csc x = -\tfrac{5}{3}$$
$$\cos x = \tfrac{4}{5} \qquad \sec x = \tfrac{5}{4}$$
$$\tan x = -\tfrac{3}{4} \qquad \cot x = -\tfrac{4}{3}$$

Again, express in terms of $\sin x$, the other five functions of x. Here

$$\sin x = \frac{\sin x}{1} = \frac{\text{opp. leg}}{\text{hyp.}}$$



$$\text{adj. leg} = \pm\sqrt{(1 - \sin^2 x)}. \quad \text{Hence}$$

$$\sin x = \sin x \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \pm\sqrt{(1 - \sin^2 x)} \qquad \sec x = \pm\frac{1}{\sqrt{(1 - \sin^2 x)}}$$

$$\tan x = \pm\frac{\sin x}{\sqrt{(1 - \sin^2 x)}} \qquad \cot x = \pm\frac{\sqrt{(1 - \sin^2 x)}}{\sin x}$$

The line definitions (using the unit circle) of the trigonometric functions should now be given. By means of these definitions the limits of the functions, as the angle approaches a multiple of a right triangle, can be illustrated geometrically in a manner most satisfactory to the student. The fundamental relations between the functions (except the reciprocal relations which follow at once from the ratio definitions) are obtained most naturally from the line definitions, as is also the proof of the addition theorem.

Circular measure should now be given with numerous examples illustrating its relations to degree measure. The student is now ready for the proof of

$$\lim_{x=0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x=0} \frac{\tan x}{x} = 1,$$

two limits of great importance in both pure and applied mathematics. Or,

THEOREM. *We may replace sinx and tanx in our calculations by x when x is a very small angle and is expressed in circular measure.*

Attention should be called to the fact that when we wish to find the functions of angles near 0° or 90°, ordinary interpolation will in general give inaccurate results, and that the above theorem should be used instead.

In the reduction of functions of any angle to the functions of an acute angle the student should be encouraged to use the forms

$$180° \pm x, \text{ or } 360° \pm x,$$

for then the name of the function remains unchanged throughout the operation and there is less liability of making a mistake. And it should be pointed out that one of the principal reasons for making such a reduction is that our tables give the values of the trigonometric functions of angles from 0° to 90° only.

In a first course the student should not be required to learn the proof of the addition theorem for any but acute angles; but his attention should be called to the fact that the theorem actually does hold true for any angles, and this statement should be illustrated by examples.

Functions of twice an angle and half an angle, in terms of the functions of the angle, should now be given. More prominence than is customary should be given to the formulas

$$\tan \frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x},$$

for they express $\tan \frac{x}{2}$ rationally in terms of $\sin x$ and $\cos x$.

In proving the four formulas usually commencing with

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

we should emphasize the important fact that they express the algebraic sum of sines or cosines in terms of their products.

Care should be taken that the student does not get tied up to any particular set of letters, Greek or Roman, as symbols for angles, or to any particular forms of the formulas. You have probably met the boy who could prove

$$\sin X-\sin Y=2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

all right, but who was completely floored by

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=\sin\alpha-\sin\beta,$$

or who could not recognize as identical the expressions

$$2\sin^2\frac{x}{2}=1-\cos x,\quad \tfrac{1}{2}-\tfrac{1}{2}\cos A=(\sin\tfrac{1}{2}A)^2.$$

A certain degree of uniformity in notation and form is desirable when teaching a beginner, but when trigonometric analysis is reached in the course it is well to make a point of freely varying both notation and form.

Now comes the derivation of the general value for all the angles having the same value of a function, the introduction of inverse trigonometric functions, and the solution of trigonometric equations.

Processes should be summarized into working rules whenever practicable. As for instance, the following:

General directions for solving a trigonometric equation.

First step. If multiple angles, fractional angles, or the sums and differences of angles are involved, reduce all to functions of a single angle, and simplify.

Second step. If the resulting expressions are not readily reducible to the same function, change all the functions into sines and cosines.

Third step. Clear of fractions and radicals.

Fourth step. Change all the functions to a single function.

Fifth step. Solve for the one function now occurring in the equation, and express the general value of the angle thus found. Only such values of the angle which satisfy the given equation are solutions.

Whether or not the graphs of the functions should be given in a first course depends on the time allowed. These graphs illustrate very vividly the property of the periodicity of the trigonometric functions. After a graph has been plotted from the calculated values of a function the student should be taught how to plot the graph by purely geometric methods from the unit circle.

As soon as the law of sines, law of cosines, and law of tangents have been derived, they should be employed in the solution of some oblique triangles making use of the natural values of the functions. Areas of triangles should also be found making use of the natural values.

And now comes the theory and use of logarithms. It is astonishing how incomplete the treatment of these topics is in almost all the current text books. Unless he has the aid of a teacher the student usually runs up against a stone wall when he comes to the application of logarithms to calculations. There are many tricks in the mathematician's trade when it comes to the use of logarithms. The texts do not always explain fully these artifices of the calculator or the peculiarities of the particular tables used. With the teacher this has become second nature and he sometimes becomes impatient at the apparent stupidity of the pupil. It is as if we expected the student to be born with an instinct for the use of logarithms! The teacher should insist on having the calculations set down according to some set form or scheme. In logarithmic computations the student should always write down an outline or skeleton of the computation before using his table at all. For, it saves time to look up all the logarithms at once and, besides, it reduces the liability of error to thus separate the theoretical part of the work from that which is purely mechanical.

All results should be verified by the student himself. The importance of this has been generally overlooked by teachers. A student gains much in interest and self-confidence when he feels independent of both his book and his teacher when it comes to proving the accuracy of his results.

It is imporiant that the student should draw the figures connected with the problems as accurately as possible. This not only leads to a better understanding of the problems themselves, but also gives a clearer insight into the meaning of the trigonometric functions and makes it possible to test roughly the accuracy of the results obtained. The only instruments necessary are a graduated ruler and a protractor, and the student should be advised to use them freely.

Throughout the subject practical problems relating to matters of common knowledge should be given as far as possible. And here, at least, we should call the attention of the student to the fact that in the examples usually given it has been assumed that the given data were exact. That is, if two sides and the inclined angle of a triangle are given, as 135 ft., 217 ft., and 25.3° respectively, we have taken for granted that these numbers are not subject to errors made in measurement. This is in accordance with the

plan followed in the problems that the student has solved in Arithmetic, Algebra, and Geometry. It should not be forgotten, however, that when we apply the principles of Trigonometry to the solution of practical problems,—engineering problems, for instance,—it is usually necessary to use data which have feen found by actual measurement and therefore are subject to error. For instance, if the length of a line is measured by a steel tape, account must be taken of the expansion due to heat as well as the sagging of the tape under various tensions. And in making several measurements one should carefully see that they are made with about the same precision. Thus, it would be folly to measure one side of a triangle with much greater care than another; for, in combining these measurements in a calculation, the result would at best be no more accurate than the worst measurement. Similarly, the angles of a triangle should be measured with the same care as the sides. The number of significant figures in a measurement are supposed to indicate the care that was intended when the measurement was made. In ordinary engineering practice only the first three or four significant figures of the measurement are not subject to error. It is therefore evident that the use of five or six place tables in calculations involving these measurements introduces an unnecessary refinement and merely adds to the labor without making the results more accurate than they would be if four place tables had been used. In a large number of cases three place tables are accurate enough.

At regular intervals the student should be required to write down from memory a list of all the fundamental formulas studied.

Brief notes on the history of the science should be given by the teacher as opportunity offers.

The computation of logarithms and of the trigonometric functions from series, De Moivre's Theorem, and the hyperbolic functions, do not properly belong to a first course in Trigonometry.

FACSIMILE EDITIONS OF JOHN BOLYAI'S SCIENCE ABSOLUTE OF SPACE.

By GEORGE BRUCE HALSTED.

At last, any one has a chance to see just how looked the most extraordinary two dozen pages in the history of human thought.

How John Bolyai himself looked, the world can never know, for before it woke up to his genius he was dead of disgust and so covered with oblivion that no picture of him remains, though his love child Dyonis, son of Rosa Orbán, still lives. His eyes were blue. So much I read in his passport

at Maros-Vásárhely, where I obtained from the college in which John's father, Farkas, was professor, its copy of the priceless gem now reproduced in facsimile by Stephen Biás de Ders (Dersi Biás István).

This facsimile is the more welcome because in my translation, now in its fourth edition, are reproduced entire in Japan in English, I put Bolyai's symbols into my own words in a terminology I had already used in translating Lobachevsky, and sacrificed to uniformity the marked distinction in notation between these two independent creators of the new universe. Both recognize that through a point A outside of a straight BC there may be an infinity of straights coplanar with BC but nowhere meeting it. This was also discovered independently by Philip Kelland, Cambridge senior wrangler and tutor of the great Sylvester, whose real name was not Sylvester but plain James Joseph. Now John Bolyai and Kelland, adhering strictly to Euclid's definition of parallels, called all these intersecting straights parallel to BC. Kelland made no further distinction among them, and his work remained sterile. The others both recognized the all-importance of a boundary line, and to these boundary lines Lobachevsky restricted the application of the word parallel; so that through every point he has two intersecting parallels to the same straight, one parallel to it toward one end of it, the other parellel to it toward the other end of it, making of parallelism a *sensed* relation. For the remaining infinity of the Bolyai and Kelland parallels Lobachevsky has no word. For them I have adopted the term *ultra-parallel.* In this terminology each parallel to a straight meets it at infinity. Two straights parallel to each other have a figurative point in common. But two straights ultra-parallel to each other, though coplanar, have not even a figurative point in common. They do not meet even at infinity. Two straights parallel in opposite senses to the same straight intersect. Two straights parallel to the same straight in the same sense are parallel to one another.

Into this terminology I translated or rather paraphrased John Bolyai, and the world has accepted it. He largely wrote in symbols of his own make, which never have been used by any one else.

His symbol for that half of the straight AB which commences at the point A and contains the point B, I translate *the ray AB.*

His first section begins: "If the ray AM is not cut by the coplanar ray BN, but is cut by every ray BP within the angle ABN, this is designated by $BN \Vdash AM$." His symbol $\Vdash$ I replaced by $\parallel$ and translate *parallel.* But for Bolyai himself it meant *asymptote.* And no one word could have expressed more of the new thought. That two rays can be asymptotes to one another seemed so strange an idea we cannot sufficiently marvel at the courage, the nerve, the fineness of our young Magyar.

B. S. Teubner, of Leipzig, has on sale 300 copies of the *Ders Facsimile.* But let those who on it see the date of the original, 1832, not forget the letter I saw at Maros-Vásárhely and was the first to publish, making the date

1823 ever memorable. On its publication thus in America Charles S. Peirce wrote in *The Nation*, March 17, 1892, p. 212, in a review of Halsted's Bolyai:

> There is a winningly enthusiastic letter from Bolyai János to his father, telling him of the great step. He says: "I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you will yourself acknowledge it. At present I can not say more than that from nothing I have created a wholly new world."

Ten years later this letter was published in Hungary in Magyar and Latin. Later came the establishment of the great Bolyai prize (Prix Bolyai) by the Hungarian Academy of Sciences, a prize of ten thousand crowns founded on the occasion of the hundredth anniversary of the birth of John Bolyai "to perpetuate the memory of this illustrious scientist," whose very grave was lost, but for one woman, who told where to erect the belated granite shaft now dedicated to dauntless genius. Before the prize was first given I ventured in *Science*, as a bit of prophecy, to predict it would go to Poincaré. It did. Now it is again to be adjudged in 1910.

Attempts are being made to fix the site of Bolyai's house, No. 1004, burned in 1876. Meanwhile the quaint house of his father Farkas, redolent with memories of Hungary's wonder-child, in which I was entertained so charmingly by Professor Koncz, who died in 1906, has been torn down to make way for a new street. Franz Schmidt of Budapesth has also passed away, who on the banks of the Danube confided to me the tales his own father told him of the fiery young Captain Bolyai, whom he saw, with his treasured Damascus sword, hack off an iron spike driven into his door-post, and who, another Ivanhoe, challenged by 13 Austrian cavalry officers at once, discomfited in succession all the challengers, a swordsman so redoubtable he could have held at bay those kindred spirits D'Artagnan, Cyrano and Captain Fracasse. Cyrano composed poetry while swording his man. Bolyai played his marvellous violin between bouts while sabreing his 13 cavalry officers. But what if he had taken it into that wonderful head of his to challenge the whole Austrian army in squads of thirteen?

His violin he expressly mentions in his will. I know not the fate of Damascus blade; but now we are given in facsimile the still more trenchant instrument wherewith he severed the bonds in which throughout the ages Euclid had held captive the mind of man.

BIANGULAR COORDINATES.

By G. B. M. ZERR, Philadelphia, Pa.

It is the purpose of this discussion to set forth a very elementary exposition of a most interesting subject* rather than anything new, with the hope that it may be further developed in future issues. In what follows the notation of Professor Genese is used.†

THE STRAIGHT LINE.

We assume two fixed points A, B as poles, and determine the position of any point P in the plane of AB by the angles $PAB=\theta$ and $PBA=\phi$, regarding θ, ϕ as positive on one side of AB and negative on the other.

Let PD be a straight line intersecting AB in D, N the foot of the perpendicular from P on AB; also let $AB=c$, $AD=d$, $BD=f$, $\cot\theta=\lambda$, $\cot\phi=\mu$.

Then $AN+BN=c$... (1).

$PN=DN\tan D=(AN-d)\tan D=NB\tan\phi$... (2).

$PN=(f-NB)\tan D=AN\tan\theta$... (3).

Eliminating AN, NB from (1), (2), (3) we get

$$f\tan D\tan\phi-c\tan\theta\tan\phi-(c-f)\tan D\tan\theta=0,$$
$$\text{or } f\tan D\tan\phi-c\tan\theta\tan\phi-d\tan D\tan\theta=0.$$

Hence $f\cot\theta-d\cot\phi-c\cot D=0$. Writing p for f, q for $-d$, r for $-c\cot D$, we get for the equation to any straight line

$$\text{I.}\quad p\lambda+q\mu+r=0.$$

Hence $\lambda=0$, $\mu=0$ represents straight lines perpendicular to AB, through A and B, respectively. Let $p_1\lambda+q_1\mu+r_1=0$ be any other line, α the angle line I makes with this last line. Then, since $r=-c\cot D$, $r_1=-c\cot D_1$,

$$\cot\alpha=\frac{(r/c)(r_1/c)+1}{r/c-r_1/c}=\frac{rr_1+c^2}{c(r-r_1)}.$$

Therefore $\cot\alpha=0$ or ∞ according as the lines are perpendicular or parallel. If $r_1=-c^2/r$, the lines are perpendicular. If $r_1=r$, the lines are parallel.

*A consideration of biangular coordinates may be found in various sources, including the following: *Quarterly Journal of Mathematics*, Volumes IX and XIII; Carr's *Synopsis of Pure Mathematics;* Milne's *Companion to Weekly Problem Papers.*

†See *Weekly Problem Papers.*

Let AB be the axis of abscissas, and the perpendicular bisector of AB, the axis of ordinates, then, it follows at once that $\lambda=(c+2x)/2y$, $\mu=(c-2x)/2y$.

Hence, $x=\frac{c}{2}(\lambda-\mu)/(\gamma+\mu)$, $y=c/(\lambda+\mu)$... (4).

The values of x and y from (4) in the general Cartesian equation gives

$$(Ac+2C)\lambda+(2C-Ac)\mu+2Bc=0.$$

This is of the form

$$p\lambda+q\mu+r=0, \text{ the same as I.}$$

(4) is used to transform from rectangular to biangular coordinates.

Let $\left.\begin{matrix} p\lambda+q\mu+r=0 \\ p\lambda_1+q\mu_1+r=0 \\ p\lambda_2+q\mu_2+r=0 \end{matrix}\right\}$ represent the same line.

Eliminating p, q, r, we get

$$\text{II.}\quad \lambda-\lambda_2=\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}(\mu-\mu_2)$$

for the equation to the line joining (λ_1, μ_1), (λ_2, μ_2).

As a simple illustration let it be required to find the locus of the vertex of a triangle when the base and the difference of the squares of the other sides are given.

Here $AB=c$, $AP^2-BP^2=m^2$. But $AP^2-BP^2=AN^2-BN^2=c(AN-BN)=m^2$. Also $\lambda=AN/PN$, $\mu=BN/PN$.

Hence, $c(\lambda-\mu)/PN=m^2$, $PN=c/(\lambda+\mu)$.

Therefore, $(c^2-m^2)\lambda=(c^2+m^2)\mu$, a straight line perpendicular to AB, and dividing it in the ratio $(c^2+m^2):(c^2-m^2)$.

The Conic Sections.

(a) By (4) the rectangular equation $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$ transforms into the biangular equation

$$b^2(4a^2-c^2)(\lambda^2+\mu^2)+2b^2(4a^2+c^2)\lambda\mu\mp4a^2c^2=0... (5).$$

When $c=2a$, (5) becomes $\lambda\mu=\pm a^2/b^2$... (6), an ellipse or hyperbola, according as we use the plus sign or the minus sign, with AB as major axis.

When $c=2ae$, (5) becomes

$$b^2(1-e^2)(\lambda^2+\mu^2)+2b^2(1+e^2)\lambda\mu\mp4a^2e^2=0,$$

$$\text{or } \lambda^2+\mu^2+\frac{2(1+e^2)}{1-e^2}\lambda\mu = \pm\frac{4a^2e^2}{b^2(1-e^2)}\ldots(7),$$

the equation for the ellipse and the hyperbola, when A, B are the foci.

If $a=b$, (6) becomes $\lambda\mu=1$ for the circle and $\lambda\mu=-1$ for the rectangular hyperbola, and (7) becomes $\lambda+\mu=0$, the line at infinity. In this last case A and B coincide and hence $c=0$. (6) is found geometrically as follows:

The geometry of the ellipse and the hyperbola gives us $PN^2:AN.NB=b^2:a^2$. As before stated, $\mu=NB/PN$, $\lambda=\pm AN/PN$.

Hence $\lambda\mu=\pm AN.NB/PN^2=\pm a^2/b^2$. From II, the secant through two points on (6) is $\lambda-\lambda_2=\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}(\mu-\mu_2)$.

But $\lambda_1\mu_1=\lambda_2\mu_2$ or $\lambda_1/\lambda_2=\mu_2/\mu_1$. Hence, $(\lambda_1-\lambda_2)/\lambda_2=-(\mu-\mu_2)/\mu_1$, or $\lambda-\lambda_2=-(\lambda_2/\mu_1)(\mu-\mu_2)$.

For the tangent at (λ_1, μ_1) we have $\lambda_1=\lambda_2$, $\mu_1=\mu_2$, and hence $\lambda-\lambda_1=-(\lambda_1/\mu_1)(\mu-\mu_1)$, or $\lambda/\lambda_1+\mu/\mu_1=2\ldots(8)$.

For the circle $\lambda_1\mu_1=1$ and the tangent is given by $\lambda\mu_1+\mu\lambda_1=2$.

(b) Let c be the distance from the focus B to the directrix A of the ellipse or the hyperbola, then if the mid-point of AB is the origin, the rectangular equation is

$$y^2+(x-\tfrac{1}{2}c)^2=e^2(x+\tfrac{1}{2}c)^2$$

which reduces to

$$4y^2+4x^2(1-e^2)-4cx(1+e^2)+c^2(1-e^2)=0\ldots(9).$$

The substitution of (4) in (9) gives $e^2\lambda^2=\mu^2+1\ldots(10)$.

This result is found geometrically as follows: P is a point on the curve, R a point on the directrix, PR is parallel to BA.

Then $BP=ePR=ePA\cos\theta$. Then $BP/PA=\sin\theta/\sin\phi=e\cos\theta$, or $\operatorname{cosec}\phi=e\cot\theta$. Hence, $1+\mu^2=e^2\lambda^2$.

For two points on this curve we easily get $e^2\lambda_1^2-\mu_1^2=e^2\lambda_2^2-\mu_2^2=1$, or $\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}=\frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}$. Hence, $\lambda-\lambda_2=\frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}(\mu-\mu_2)\ldots(11)$, is the secant through (λ_1, μ_1), (λ_2, μ_2). For the tangent $\lambda_1=\lambda_2$, $\mu_1=\mu_2$.

Hence, $\lambda-\lambda_1=\frac{\mu_1}{e^2\lambda_1}(\mu-\mu_1)$ is the tangent at (λ_1, μ_1). This further reduces to $e^2\lambda\lambda_1-\mu\mu_1=e^2\lambda_1^2-\mu_1^2=1$. That is, $e^2\lambda\lambda_1-\mu\mu_1=1$ is the tangent.

(c) If AB is a chord of a circle, P the angle inscribed opposite AB, then $\theta+\phi=\pi-P$, $\cot(\theta+\phi)=-\cot P$.

Hence $\lambda\mu+(\lambda+\mu)\cot P=1$ is the equation to the circum-circle of APB.

If $P=\frac{1}{2}\pi$, $\cot P=0$, and $\lambda\mu=1$, as before stated.

(d) Let $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$ be the rectangular equation for the general conic. Substituting the values of x and y from (4) this reduces to the form

$$\text{III.}\quad a\lambda^2+2h\lambda\mu+b\mu^2+2g\lambda+2f\mu+e=0.$$

The tangent to this conic at the point (λ_1, μ_1) is

$$\text{IV.}\quad a\lambda_1\lambda+h(\lambda_1\mu+\mu_1\lambda)+b\mu_1\mu+g(\lambda+\lambda_1)+f(\mu+\mu_1)+e=0.$$

If $a=b=0=g=f$, III becomes $\lambda\mu$+a constant=0; IV becomes $\lambda_1\mu+\mu_2\lambda$ +a constant=0.

If $a=b=0$, we get the conic referred to any chord. Its equation is of the form

$$\text{V.}\quad \lambda\mu+g_1\lambda+f_1\mu+e_1=0.$$

Its tangent at (λ_1, μ_1) is of the form

$$\text{VI.}\quad \lambda_1\mu+\mu_1\lambda+g_1(\lambda+\lambda_1)+f_1(\mu+\mu_1)+e_1=0.$$

V and VI can be written as follows:

$$(\lambda+f_1)(\mu+g_1)=\text{a constant},$$
$$(\lambda+f_1)(\mu_1+g_1)+(\lambda_1+f_1)(\mu+g_1)=\text{a constant}=2(\lambda_1+f_1)(\mu_1+g_1).$$

Hence, $\dfrac{\lambda+f_1}{\lambda_1+f_1}+\dfrac{\mu+g_1}{\mu_1+g_1}=2$ is a neater form for VI.

An Application.

(e) As an application of biangular coordinates, suppose we have given the base AB of a triangle and the locus of the vertex, to find the locus of the symmedian point K.

Let the vertex C describe the straight line, $p\cot A+q\cot B+r=0$... (13).

Let $\cot A=P$, $\cot B=Q$. Since the median AM and the symmedian AK are equally inclined to the bisector of A,

$$\angle MAB+\angle KAB=\angle A, \text{ or } \cot KAB=\lambda=\frac{\cot A\cot MAB+1}{\cot MAB-\cot A},$$

$$\text{and } \lambda=\frac{2P^2+PQ+1}{P+Q},\quad \mu=\frac{2Q^2+PQ+1}{P+Q}.$$

$$(P+Q)(\lambda-\mu)=2(P^2-Q^2), \text{ or } P=\frac{\lambda-\mu}{2}+Q,\quad Q=P-\frac{\lambda-\mu}{2}.$$

The value of Q substituted in the value of λ gives

$$6P^2-(5\lambda-\mu)P=\mu\lambda-\lambda^2-2, \text{ or } P=\frac{5\lambda-\mu\pm\sqrt{(\lambda^2+14\lambda\mu+\mu^2-48)}}{12}.$$

$$\text{Similarly, } Q=\frac{5\mu-\lambda\pm\sqrt{(\lambda^2+14\lambda\mu+\mu^2-48)}}{12}.$$

These values of P and Q substituted in (13) gives for the required locus

$$[(5p-q)\lambda+(5q-p)\mu+12r]^2=(p+q)^2)(\lambda^2+14\lambda\mu+\mu^2-48),$$
$$\text{or } (2p^2-pq)\lambda^2-2(p^2+q^2-pq)\lambda\mu+(2q^2-pq)\mu^2$$
$$+2r(5p-q)\lambda+2r(5q-p)\mu+12r^2+4(p+q)^2=0.$$

That is, $a\lambda^2+2\lambda\mu h+b\mu^2+2g\lambda+2f\mu+e=0$, which is the equation to the general conic.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

327. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

The coefficients of the algebraical equation $f(x)=0$ are all integers. Show that if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let the equation be $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$.

Then by hypothesis, a_n, and also $(a_0+a_1+a_2+\ldots+a_{n-1}+a_n)$ are odd numbers. By subtraction, $(a_0+a_1+\ldots+a_{n-1})$ is an even number; hence, either all these coefficients are even numbers, or there is an even number of odd coefficients.

Suppose (I) that $a_0, \ldots, a_{n-1}$ are all even. Then the substitution for x of *any* integer will give an even result for these terms, which cannot combine with the odd a_n to vanish.

Suppose (II) that an even number of coefficients $a_0, \ldots, a_{n-1}$ are odd integers. The substitution of any even integer for x cannot satisfy the equation, just as in (I). Any power of an odd integer (substituted for x) is odd, so that for each odd coefficient, one odd term is obtained. But there is an even number of these, besides a_n. Thus it is again impossible to have the equation satisfied by an integer.

328. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

If $x^2+xy+y^2=3a^2$, find the maximum value of $bx+cy$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

$x^2+xy+y^2=3a^2$, $bx+cy$=maximum.

$$\therefore \frac{dx}{dy}=-\frac{x+2y}{2x+y}=-\frac{c}{b}, \text{ or } y=\frac{(2c-b)x}{2b-c}.$$

$$\therefore x^2=\frac{a^2(2b-c)^2}{b^2-bc+c^2}; \text{ and } x=\pm\frac{a(2b-c)}{\sqrt{(b^2-bc+c^2)}}, \quad y=\pm\frac{a(2c-b)}{\sqrt{(b^2-bc+c^2)}}.$$

$$\therefore bx+cy=2a\sqrt{(b^2-bc+c^2)}=2a\sqrt{\frac{b^3+c^3}{b+c}} \text{ is a maximum.}$$

Also solved by F. L. Griffin and S. G. Barton.

329. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Between the quantities a and b there are inserted n arithmetical and n harmonical means, and a series of n terms is formed by dividing each arithmetical by the corresponding harmonical mean. Show that the sum of the series is, $n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right]$.

Solution by HOWARD C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb., and S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

The n arithmetic means between a and b are:

$$(1)\quad \frac{b+na}{n+1}, \quad \frac{2b+(n-1)a}{n+1}, \ldots, \quad \frac{rb+(n-r+1)a}{n+1}, \ldots$$

The n harmonic means between a and b are:

$$(2)\quad \frac{ab(n+1)}{nb+a}, \quad \frac{ab(n+1)}{(n-1)b+2a}, \ldots, \quad \frac{ab(n+1)}{(n-r+1)b+(r-1)a}, \ldots$$

Dividing the terms of (1) by the corresponding terms of (2),

$$\frac{(b+na)(a+nb)}{ab(n+1)^2}, \quad \frac{[2b+(n-1)a][(n-1)b+2a]}{ab(n+1)^2}, \ldots,$$

$$\frac{[rb+(n-r+1)a][(n-r+1)b+2a]}{ab(n+1)^2}, \ldots, \text{the series to be summed.}$$

Hence $S=\frac{ab(n^2+1)+n(a^2+b^2)}{ab(n+1)^2}+\frac{ab[(n-1)^2+r^2]+2(n-1)(a^2+b^2)}{ab(n+1)^2}+\ldots$

$$+\frac{ab[(n-r+1)^2+r^2]+r(n-r+1)(a^2+b^2)}{ab(n+1)^2}+\ldots$$

$$=\frac{1}{ab(n+1)^2}\{ab[n^2+(n-1)^2+\ldots+1^2]+ab[1^2+2^2+\ldots+n^2]$$

$$(a^2+b^2)[n+2(n-1)+3(n-2)+\ldots+r(n-r+1)+\ldots+n]\}=$$

$$=\frac{1}{ab(n+1)^2}\left[2ab\frac{n(n+1)(2n+1)}{6}+(a^2+b^2)\left(\frac{n(n^2+1)}{2}-\frac{n(n^2-1)}{3}\right)\right]$$

$$=\frac{1}{ab(n+1)^2}\left[2ab\frac{n(n+1)(2n+1)}{6}+(a^2+b^2)\left(\frac{n(n+1)(n+2)}{6}\right)\right]$$

$$=\frac{n}{ab(n+1)^2}\left[\frac{6ab(n+1)^2}{6}-\frac{n^2+3n+2}{6}\cdot 2ab+\frac{(n+1)(n+2)}{6}(a^2+b^2)\right]$$

$$=n\left[1+\frac{a^2-2ab+b^2}{6ab(n+1)^2}\cdot(n+1)(n+2)\right]=n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right].$$

Solved similarly by G. B. M. Zerr, and J. Scheffer.

GEOMETRY.

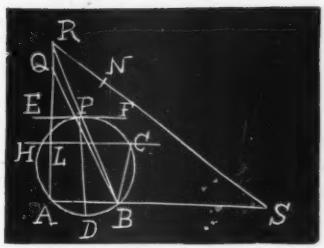
352. Proposed by G. I. HOPKINS, Professor of Astronomy, High School, Manchester, N. H.

Required, to construct the triangle, having given the base, vertical angle and sum of the altitude and the two remaining sides.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $AB=a$ be the given base; ACB the given vertical angle; p=to the sum of the altitude and the remaining sides. On AB describe the seg-

ment APB containing the given angle. Draw the diameter PD perpendicular to AB. Also draw AQ perpendicular to AB. Draw BP meeting AQ in Q. With B as a center, and a radius equal to p, describe an arc cutting AQ produced in R. With R as a center, and a radius equal to $p+AQ$, describe an arc cutting AB produced in S. On SR measure off SN=to SA. Then RN =the altitude. Take $AL=RN$, and draw HLC parallel to AB, cutting the circle in C. Draw AC, BC. Then ACB is the required triangle. For let x, y, z be the sides BC, AC, and the altitude. Then $xy\sin C=az$, $x+y+z=p$, $a^2=x^2+y^2-2xy\cos C$.



$\therefore a^2+2xy(1+\cos A)=(p-z)^2$. $\therefore z^2-2(p+a\cot\frac{1}{2}C)=a^2-p^2$.

$\therefore z=p+a\cot\frac{1}{2}C-\sqrt{[(p+a\cot\frac{1}{2}C)^2-(p^2-a^2)]}$.

$\angle AQB=\frac{1}{2}C$, $AQ=a\cot\frac{1}{2}C$, $RS=p+a\cot\frac{1}{2}C$, $AR=\sqrt{(p^2-a^2)}$, $AS=\sqrt{[(p+a\cot\frac{1}{2}C)^2-(p^2-a^2)]}$.

$\therefore RN=z$. $\therefore$ The triangle ACB contains all the required parts.

Also solved by J. Scheffer.

353. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

In a given circle place two chords which shall be in a given ratio and also a given distance apart.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let the ratio $m:n$ to distance apart$=d$; the radius of the given circle $=r$; u, v the distances of the chords from the center.

Then $2\sqrt{(r^2-u^2)}$, $2\sqrt{(r^2-v^2)}$ are the lengths of the chords.

$\therefore m\sqrt{(r^2-v^2)}=n\sqrt{(r^2-u^2)}$, or $(m^2-n^2)r^2=m^2v^2-n^2u^2$, and $u+v=d$.

$$\therefore u=\frac{m^2d-\sqrt{[r^2(m^2-n^2)^2+m^2n^2d^2]}}{m^2-n^2},$$

$$v=\frac{\sqrt{[r^2(m^2-n^2)^2+m^2n^2d^2]}-n^2d}{m^2-n^2}.$$

Hence, if $AB=d$, take $AC=u$, $CB=v$, and with C as center and radius r describe a circle, through A and B perpendicular to AB draw lines intersecting this circle. The chords of the circle formed by these lines are the chords required.

Also solved by S. A. Corey.

354. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Find the condition that triangles which are circumscribed to one of two confocal parabolas may be inscribed in the other.

Solution by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Let $y^2=4a(x+a)$, $(y\cos\beta+x\sin\beta)^2=4A(x\cos\beta-y\sin\beta+A)$ be the confocal parabolas.

(1) $y^2-4ax-4a^2=0$.
(2) $y^2\cos^2\beta+x^2\sin^2\beta+2xy\sin\beta\cos\beta-4Ax\cos\beta+4Ay\sin\beta-4A^2=0$.

Calculating the invariants for (1) and (2) we get

$$\Delta=-4a^2, \quad \Theta=-4a(a+2A\cos\beta),$$
$$\Theta'=-4A(A+2a\cos\beta), \quad \Delta'=-4A^2.$$

The condition is given by $\Theta^2=4\Delta\Theta'$.

$\therefore 16a^2(a+2A\cos\beta)^2=16a^2A(A+2a\cos\beta)$.

$\therefore a^2/A^2+2(a/A)\cos\beta+4\cos^2\beta=1$, $a/A=-\cos\beta\pm\sqrt{(1-3\cos^2\beta)}$.

$\therefore \cos\beta>1/\sqrt{3}$.

$\therefore\beta$ lies between 54° 44′ and 125° 16′, and also between 234° 44′ and 305° 16′.

If $\beta=\frac{1}{3}\pi$, $a=\infty A$ or $-A$.
If $\beta=\frac{1}{2}\pi$, $a=A$ or $-A$.
If $\beta=\frac{2}{3}\pi$, $a=A$ or ∞A, etc.

CALCULUS.

282. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

A rectangular beam of length l and width w is taken horizontally from a hall of width b into a corridor at right angles to the hall. Find the width of the smallest corridor into which it can be taken.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and J. M. ARNOLD, Crompton, R. I.

Let $ABCD$ be the beam, the corner A against the hall wall, the corner B against the corrider wall; the point P of the beam against the corner of of meeting of hall and corrider.

Let $AB=l$, $BC=w$, $x=$width of corrider, PQ the portion of the width the corrider under the beam, QR the remainder of the width.

Then $PQ+QR=x$, $AQ+QB=l$, $ER=b$, $\angle BAE=\theta$. Then $PQ=w\operatorname{cosec}\theta$, $AQ=b\operatorname{cosec}\theta$, $QR=(l-b\operatorname{cosec}\theta)\cos\theta$.

$\therefore x=w\operatorname{cosec}\theta+(l-b\operatorname{cosec}\theta)\cos\theta$... (1).

Differentiating (1) we get, $w\operatorname{cosec}\theta\cot\theta+l\sin\theta=b\operatorname{cosec}^2\theta$.

$\therefore w\cos\theta=b-l\sin^3\theta$... (2).

The value of θ from (2) in (1) gives the width x required.

(2) becomes $l^2\sin^6\theta-2bl\sin^3\theta+w^2\sin^2\theta+b^2-w^2=0$.

Also solved by J. E. Sanders and the Proposer.

283. Proposed by B. F. FINKEL, Ph. D., Drury College.

By means of the calculus, determine the angle of minimum deviation of a ray of monochromatic light in passing through a triangular prism.

Solution by C. N. SCHMALL, New York City.

We have, with the usual notation,

$$\sin\phi=\mu\sin\phi'\ldots(1),$$
$$\sin\psi=\mu\sin\psi'\ldots(2),$$
$$\phi'+\psi'=\text{a constant, } k, \text{ say}\ldots(3),$$
$$u=\phi+\psi=\text{a minimum}\ldots(4).$$

Solving (1) and (2), we have, $\phi'=\sin^{-1}(\sin\phi/\mu)$, $\psi'=\sin^{-1}(\sin\psi/\mu)$. Substituting in (3), we have

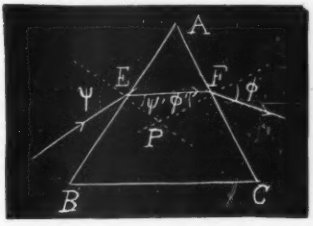
$$\sin^{-1}\left(\frac{\sin\phi}{\mu}\right)+\sin^{-1}\left(\frac{\sin\psi}{\mu}\right)=k\ldots(5).$$

Differentiating with respect to ϕ and reducing slightly, we have

$$\frac{\cos\phi}{\sqrt{(\mu^2-\sin^2\phi)}}+\frac{\cos\psi}{\sqrt{(\mu^2-\sin^2\psi)}}\frac{d\phi}{d\psi}=0\ldots(6).$$

Differentiating (4), we have

$$\frac{du}{d\phi}=1+\frac{d\psi}{d\phi}=0\ldots(7).$$



Equating the values of $\frac{d\psi}{d\phi}$, found from (6) and (7), we have finally,

$$\frac{\cos^2\phi}{\mu^2-\sin^2\phi}=\frac{\cos^2\psi}{\mu^2-\sin^2\psi}. \quad \therefore\phi=\psi.$$

Now, the deviation is $D=(\psi-\psi')+(\phi-\phi')=2(\phi-\phi')$, (since $\psi=\phi$)...(8).

But this will not be the case unless the $\triangle ABC$ is isosceles. Then, if θ be the angle of the prism, that is, angle A, we have

$$\phi' = \tfrac{1}{2}\theta, \text{ and from (1), } \mu = \frac{\sin\phi}{\sin\phi'} \ldots (9),$$

but from (8), $\phi = \frac{1}{2}(D+2\phi') = \frac{1}{2}(D+\theta)$. Hence, substituting in (9), we have,

$$\mu = \frac{\sin\frac{1}{2}(D+\theta)}{\sin\frac{1}{2}\theta} \ldots (10),$$

which gives the relation between the minimum deviation D, the angle of the prism θ, and the index of refraction μ.

Also solved by G. B. M. Zerr and J. Scheffer.

284. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

Inscribe the triangle of maximum area in a given circle.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let the diameter, of length d, through any vertex make angles α and β with the two sides of the triangle meeting there. Then two sides of the triangle are $d\cos\alpha$ and $d\cos\beta$; and their included angle is $\alpha+\beta$. [$\alpha-\beta$ is excluded, since a triangle contained in a semi-circle clearly can not be the maximum.] Now the area is $\frac{1}{2}(d\cos\alpha).(d\cos\beta).\sin(\alpha+\beta)$; so that we have to render a maximum the function $F(\alpha, \beta) \equiv \cos\alpha.\cos\beta.\sin(\alpha+\beta)$.

$$\text{But } \frac{\partial F}{\partial \alpha} = \cos\beta\,(\cos\alpha.\cos\overline{\alpha+\beta} - \sin\alpha.\sin\overline{\alpha+\beta}) = \cos\beta.\cos(2\alpha+\beta),$$

$$\frac{\partial F}{\partial \beta} = \cos\alpha.\cos(2\beta+\alpha).$$

Equating these to zero, we obtain since $\cos\alpha \neq 0$, $\cos\beta \neq 0$ if there is actually a triangle:

$$2\alpha+\beta = \tfrac{1}{2}\pi, \quad 2\beta+\alpha = \tfrac{1}{2}\pi, \quad \text{whence } \alpha = \beta = \tfrac{1}{6}\pi.$$

$$\text{Also, } \frac{\partial^2 F}{\partial \alpha^2} = -2\sin(2\alpha+\beta)\cos\beta, \quad \frac{\partial^2 F}{\partial\alpha\,\partial\beta} = -\sin\overline{2\alpha+2\beta},$$

$$\frac{\partial^2 F}{\partial \beta^2} = -2\sin(2\beta+\alpha).\cos\alpha; \text{ whence, } \left[\frac{\partial^2 F}{\partial\alpha^2}\cdot\frac{\partial^2 F}{\partial\beta^2} - \left(\frac{\partial^2 F}{\partial\alpha\,\partial\beta}\right)^2\right]_{\alpha=\beta=\frac{1}{6}\pi} = \frac{3}{2},$$

the positive result together with the negative value of $\partial^2 F/\partial\alpha^2$ showing a maximum value of $F(\alpha, \beta)$.

For $\alpha=\beta=\frac{1}{6}\pi$, the sides $d\cos\alpha$, $d\cos\beta$ are equal, and the isosceles triangle is equilateral, having its vertex angle equal to $\frac{1}{3}\pi$.

Also solved by H. C. Feemster, Wilmer Thompson, G. B. M. Zerr, and C. N. Schmall.

285. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

If R_1 and R_2 are the radii of curvature of an ellipse at the extremities of a pair of conjugate diameters, show that $R_1^{2/3}+R_2^{2/3}=\frac{a^2+b^2}{(ab)^{2/3}}$, where a, b, are the semi-axes.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $x^2/a^2+y^2/b^2=1$ be the ellipse. Also let $x=a\cos\theta$, $y=b\sin\theta$.

$$dx=-a\sin\theta\,d\theta,\quad d^2x=-a\cos\theta\,d\theta^2-a\sin\theta\,d^2\theta,$$
$$dy=b\cos\theta\,d\theta,\quad d^2y=-b\sin\theta\,d\theta^2+b\cos\theta\,d^2\theta.$$

$$\therefore R_1=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y-dy\,d^2x}=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}.$$

$$R_2=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}}{ab}.\quad\text{But }\phi=\theta+\tfrac{1}{2}\pi.$$

$$\therefore R_2=\frac{(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{3}{2}}}{ab}.$$

$$R_1^{2/3}+R_2^{2/3}=\frac{(a^2+b^2)\,(\sin^2\theta+\cos^2\theta)}{(ab)^{2/3}}=\frac{a^2+b^2}{(ab)^{2/3}}.$$

II. Solution by HOWARD C. FEEMSTER, Professor of Mathematics, York College, York, Neb.; S. G. BARTON, Professor of Mathematics, Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, Hagerstown, Md.

Let x_1, y_1 be the point of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with a diameter, and $-\frac{ay_1}{b}$, $\frac{bx_1}{a}$, the point of intersection of the ellipse with the diameter conjugate to the first.

But the radius of curvature of any point on a curve equals:

$$(1)\quad \rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

For the ellipse,

$$(2)\ \frac{dy}{dx}=-\frac{b^2x}{a^2y},\ \text{and}\ (3)\ \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

From (1), (2), and (3), making the sign positive,

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

Hence, for (x_1, y_1), $\rho=R_1=\frac{(a^4y_1^2+b^4x_1^2)^{\frac{3}{2}}}{a^4b^4}$.

And for $-\frac{ay_1}{b}$, $\frac{bx_1}{a}$, $\rho=R_2=\frac{(a^2b^2x_1^2+a^2b^2y_1^2)^{\frac{3}{2}}}{a^4b^4}$.

$$\therefore R_1^{\frac{2}{3}}+R_2^{\frac{2}{3}}=\frac{(a^2+b^2)\ (b^2x_1^2+a^2y_1^2)}{(ab)^{\frac{4}{3}}}=\frac{a^2+b^2}{(ab)^{\frac{2}{3}}}.$$

MECHANICS.

235. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

A uniform heavy rod turns freely round a hinge at one end and rests with the other against a rough vertical wall, at angle, α, to the wall. Find the angle of arc on which this end may rest, and the pressures at the ends of the arc.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $2a$=length of rod; w, its weight; O, the hinge; OBC, the plane of the rod (vertical) perpendicular to the wall; α=angle BOC; μ=coefficient of friction between rod and wall; OA, the position of the rod at any time; OD, the projection of OA on the plane OBC; θ=angle DOA; ϕ=angle DOC; R=reaction or pressure at wall; OA, the axis of x; z, normal to AOD; y perpendicular to both x and z.

Then $OD=2a\cos\theta$, $OC=2a\sin\alpha$, $\phi=\cos^{-1}(\sin\alpha/\sin\theta)$.

The weight of the rod acting at its mid-point is equivalent to $w\cos\phi$ parallel to z, and $w\sin\phi$ parallel to OD. R is perpendicular to both OA and AD. Taking moments about y and z, respectively, we have, $aw\cos\phi=2aR$, $aw\sin\phi\sin\theta=2a\mu R$.

$\therefore \sin\theta=\mu\cot\theta=\mu\sin\alpha/\sqrt{(\cos^2\theta-\sin^2\alpha)}$.

$\therefore \sin^2\theta=\frac{1}{2}[\cos^2\alpha\pm\sqrt{(\cos^4\alpha-4\mu^2\sin^2\alpha)}]=P$. Use positive sign.

$R=(w/2)\cos\phi=[w\sin\alpha]/[2\sqrt{(1-P)}]$.

236. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

A simple beam length $2a$, supported at the ends, is loaded with c pounds per running foot at the ends and increases uniformly to the center, where it is b pounds per running foot. Find deflection at center due to this load.

Solution by J. E. SANDERS, Weather Bureau, Chicago, Ill., and the PROPOSER.

The load is represented by a double trapezoid, bases b, c; altitude a for each. Let $AB=2a$, $AF=FB=a$, $AE=BC=c$, $DF=b$, G the place of any point distant $AG=x$ from the left end. Draw GH parallel to AE, EK parallel to AG. Then $EK=AG=x$, $GK=AE=c$, $HG=c+[(b-c)x]/a$.

Area $AGHE=x(2ac+bx-cx)/2a$, area $ABCDE=a(b+c)$, area $AGKE=cx$, area $EKH=(bx^2-cx^2)/2a$.

Let z=distance of center of gravity of $AGHE$ from GH.

Then $z(\text{area } AGHE)=\frac{1}{2}x(\text{area } AGKE)+\frac{1}{3}x(\text{area } EKH)$.

$\therefore zx(2ac+bx-cx)/2a=\frac{1}{2}cx^2+(bx^3-cx^3)/6a$.

$$\therefore z=\frac{3acx+bx^2-cx^2}{3(2ac+bx-cx)}.$$

Bending moment$=\frac{1}{2}a(b+c)x-z(\text{area } AGHE)$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-\frac{1}{6}(3acx^2+bx^3-cx^3)/a$.

$$EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}cx^3-\frac{bx^4}{24a}+\frac{cx^4}{24a}+C.$$

When $x=a$, $dy/dx=0$. $\therefore C=-\frac{5}{24}a^3b-\frac{1}{8}a^3c$.

$$\therefore EIdy/dx=\tfrac{1}{4}abx^2+\tfrac{1}{4}acx^2-\tfrac{1}{6}cx^3-\frac{bx^4}{24a}+\frac{cx^4}{24a}-\tfrac{5}{24}a^3b-\tfrac{1}{8}a^3c.$$

$$\therefore EIy=\tfrac{1}{12}abx^2+\tfrac{1}{12}acx^2-\tfrac{1}{24}cx^4-\frac{bx^5}{120a}+\frac{cx^5}{120a}-\tfrac{5}{24}a^3bx-\tfrac{1}{8}a^3cx.$$

No constant being added as x and y are zero together.

When $x=a$, $EIy=-\frac{2}{15}a^4b-\frac{3}{40}a^4c$. If $c=0$, $EIy=-\frac{2}{15}a^4b$. If $c=b$, $EIy=-\frac{5}{24}a^4b$.

If the load consisted of b pounds per running foot at the ends and decreased to c pounds per running foot at the middle, then z times area $AGHE=\frac{1}{2}x^2.HG+\frac{2}{3}x(\text{area } EKH)$ where $AE=b$ and K is on AE instead of GH as before; $HG=(ab-bx+cx)/a$; area $AGHK=(abx-bx^2+cx^2)/a$.

$$z=\frac{3abx-bx^2+cx^2}{3(2ab-bx+cx)},$$ the same as above by replacing c with b.

$\therefore$ Bending moment$=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$$EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}(bx^3-\frac{bx^4}{4a}+\frac{cx^4}{4a})+C.$$

When $x=a$, $dy/dx=0$; $\therefore C=-\frac{1}{8}a^3b-\frac{5}{24}a^3c$.

$$\therefore EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}bx^3+\frac{bx^4}{24a}-\frac{cx^4}{24a}-\tfrac{1}{8}a^3b-\tfrac{5}{24}a^3c.$$

$$EIy=\tfrac{1}{12}abx^3+\tfrac{1}{12}acx^3-\tfrac{1}{24}bx^4+\frac{bx^5}{120a}-\frac{cx^5}{120a}-\tfrac{1}{8}a^3bx-\tfrac{5}{24}a^3cx.$$

No constant being added.

When $x=a$, $EIy=-\frac{3}{40}a^4b-\frac{2}{15}a^4c$.

If $c=0$, $EIy=-\frac{3}{40}a^4b$. If $b=c$, $EIy=-\frac{5}{24}a^4b$.

y is the deflection required in each case.

Also solved by S. G. Barton.

PROBLEMS FOR SOLUTION.

ALGEBRA.

333. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the infinite series

$$\frac{1}{(m+1)^2}+\frac{(2m-1)}{(2m+1)^2}+\frac{(3m-1)^2}{(3m+1)^4}+\frac{(4m-1)^3}{(4m+1)^5}+\frac{(5m-1)^4}{(5m+1)^6}+\ldots$$

334. Proposed by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Sum the series, $2^n-n\,2^{n-2}+\frac{n(n-3)}{2!}2^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-6}$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\frac{n(n-6)(n-7)(n-8)(n-9)}{5!}2^{n-10}+\ldots$$

GEOMETRY.

362. Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

363. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given, base, vertical angle, and difference between altitude and sum of the other two sides.

364. Proposed by R. C. ARCHIBALD, Providence, R. I.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. ["Euclidean constructions" are particularly desired.]

CALCULUS.

290. Proposed by C. N. SCHMALL, New York City.

When the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, represents an ellipse, show (by integration) that its area is

$$\frac{\pi\,(af^2+bg^2+ch^2-abc-2fgh)}{(ab-h^2)^{\frac{3}{2}}}.$$

291. Proposed by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

Integrate $\frac{dy}{dx}=ay^2+bx^m$.

MECHANICS.

244. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A load P is supported by three strings of equal size lying in the same plane. The middle string is vertical, one string makes with it the angle θ on one side, and the second string makes with it the angle ϕ on the other side. Find the stresses in the strings.

245. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A body moves with constant velocity in the circumference of an ellipse. Find the rate of approach (1) to the center, (2) to one of the foci, for any point in the ellipse.

NOTES AND NEWS.

It is desired to include in this column notes of interest concerning the activities of teachers of college mathematics in the field of education. The editors will gladly receive news of this kind from any source. Such items may be transmitted to any one of the three editors.

Columbia University announces that several new officers of instruction will be added to the Department of Mathematics at the close of the present academic year.

Professor S. C. Newson, of the Department of Mathematics in the University of Kansas, died suddenly at his home on Thursday evening, February 17, 1910.

Professor Granville's paper in the present number is the second in the series on the teaching of college mathematics. The next paper in this series will be by Professor H. L. Rietz, of the University of Illinois, on the Teaching of College Algebra.

A series of five lectures on graphical methods in the teaching of mathematics will be given at the University of Michigan during February, by Professor Carl Runge of Goettingen, Germany, who is Kaiser Wilhelm Exchange Professor at Columbia University during the present year. The titles include: Methods of graphical calculation; Graphical representation of functions, Graphical integration and differentiation, Graphical treatment of differential equations.

Mr. R. M. Ginnings, formerly of Kirksville Normal School, Missouri, is professor of mathematics in the State Normal School at Macomb, Illinois. Mr. Ginnings had recently completed a year's graduate study, on leave of absence from Kirksville, and had attained the Master's degree in mathematics.

Professor R. L. Rhoton of Georgetown College, Kentucky, is chairman of a committee of the State Teachers' Association to revise the course of study in mathematics in the Schools of Kentucky.

Mr. E. E. Whitford, Tutor in Mathematics in the College of the City of New York, is the author of an article in the January number of *School Science and Mathematics* entitled: "An American Syllabus in Algebra," in which the results of the various algebra syllabi of recent years are summed up in tabular form for convenient reference.

Professor W. A. Manning, of Stanford University, is exchange professor at the University of Illinois during the present year.

Professor H. L. McAlister, of Ouachita College, was a member of the committee of five of the State Teachers' Association of Arkansas, by whom was prepared the recently published outline of a four years' course in high school mathematics. The algebra section of this outline is a reproduction of the report on algebra published by the Central Association of Science and Mathematics in 1907, which has had such a wide circulation and influence during the past three years.

BOOKS.

Rara Arithmetica. A catalog of the Arithmetics written before the year MDCI, with a description of those in the Library of George Arthur Plimpton of New York. By David Eugene Smith, of Teachers' College, Columbia University. Large 8vo. Cloth. xiii+507 pages. Price, $5.50. Boston and Chicago: Ginn & Co.

This work is of great value to the teacher of elementary mathematics, and should be eagerly sought after by those who are teaching arithmetic. One needs to know the evolution of every subject he is teaching, and never before has there been given such an exhaustive description of the arithmetics used prior to 1601.

The book abounds with facsimile reproductions of title pages and figures.

The publishers have spared no pains to make the book neat and attractive. Their undertaking in the production of books that are of inestimable value to the scholar and investigator, but of little commercial value, should be encouraged by a large demand for such books as the one before us. F.

Ganot's Physics. Seventeenth Edition, Revised to 1905. By A. W. Reinhold, M. A., F. R. S., Professor of Physics, Royal Naval College, Greenwich, England. 8vo. Cloth, xi+1169 pages. Price, $5.00. New York: William Wood & Co.

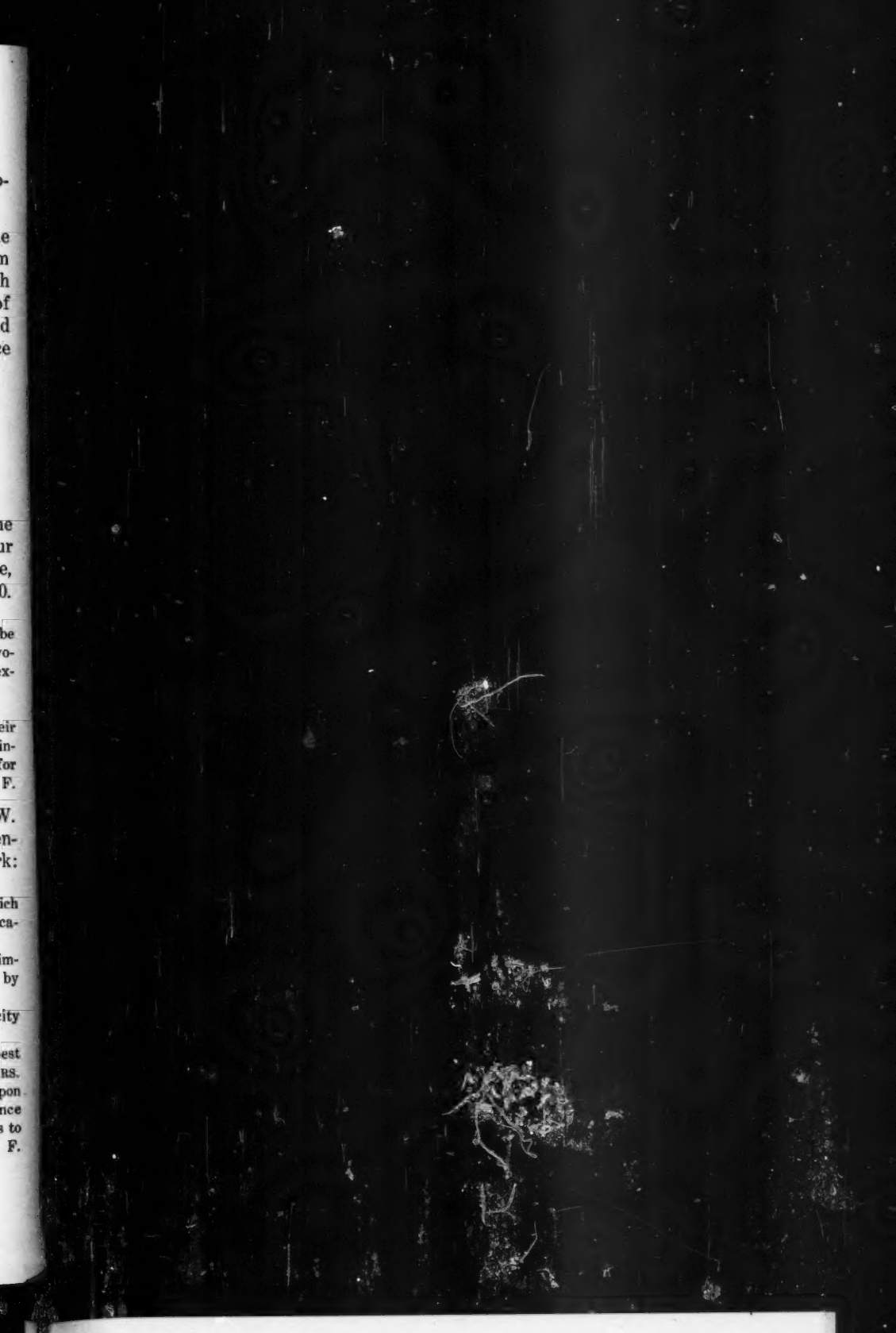
In the present edition, while the general character and scope of the work, which marked the Sixteenth Edition have been maintained, considerable additions and modifications in detail have been made, and the whole has been carefully revised.

The chapter on the Steam Engine has been rewritten and brought up-to-date; a similar service has also been performed in connection with the articles on Photography by competent authorities.

Numerous additional examples and problems especially in Magnetism and Electricity will be found at the end of the volume.

Professors of some of the largest universities in America, consider it the best all around text-book on this subject published. PUBLISHERS.

The typogoaphy of this book is very good, and its engravings have been drawn upon by many of the foremost authors who have written texts on Physics since the appearance of the first edition of Ganot's. The seventeenth edition is a splendid work and deserves to be popular. F.



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THE TEACHING OF COLLEGE ALGEBRA.

By H. L. RIETZ, University of Illinois.

1. Introduction. The expression "College Algebra" is used in this paper to designate the algebra which should be taught to freshmen students who have met the entrance requirements of one and one half years of algebra in the secondary schools. This expression is here employed because it has come into rather general use in this country for the course given in the first college year, and not with any thought that it is deemed adequate algebraic training to be expected from a college graduate interested in mathematics. On this point, it is to be understood that a much more advanced course, which includes such subjects as the theory of linear dependence and the theory of matrices, should be taken late in the college course.

The question of the time at our disposal may considerably modify the character of the course in college algebra. In what follows, the writer assumes, as a minimum of time available for college algebra, that time which should be apportioned to algebra if trigonometry, algebra, and analytic geometry together are to constitute a course with five class periods per week through a college year.

2. Selection of material. The chief danger in the selection of material and presentation of college algebra is that it is likely to be a sort of scrap heap of disconnected or rather remotely connected topics, rather than an organized body of knowledge. While this is a danger, it is not a necessary characteristic of the course. The number of subjects classed under college algebra is usually too large for the time allotted to the course. The presentation of so many subjects is due, in the main, to the call from branches of mathematics higher up for a working knowledge on the part of the student of various algebraic methods; and, certainly, one criterion for judging the importance of a subject is its use in other branches of mathematics. It is not undesirable to treat a large number of topics, provided the time is available, and if there are a few underlying principles or a method of attack by which the topics are unified or brought into relation with each other. In fact, it is familiarity with a large variety of topics and their inter-relations that should be sought in elementary college courses in mathematics.